Problem Sheet 11

Exercise 11.1

Prove that a sequence of probability measures (μ_n) on $C([0,T];\mathbb{R})$ is tight if and only if both

$$\lim_{\lambda\to\infty}\sup_{n\in\mathbb{N}}\mu_n\left(\{f:|f(0)|>\lambda\}\right)=0 \tag{1}$$

and for all $\varepsilon>0$,

$$\lim_{\delta\to 0}\sup_{n\in\mathbb{N}}\mu_n\left(\left\{f:\max_{|t-s|<\delta}|f(t)-f(s)|>\varepsilon\right\}\right)=0. \tag{2}$$

Hint: Recall the following version of Arzelá-Ascoli: The closure of a set $A\subset C([0,T];\mathbb{R})$ is relatively compact if and only if both

$$\sup_{f\in A}|f(0)|<\infty$$

and

$$\lim_{\delta\to 0}\sup_{f\in A}\max_{|t-s|<\delta}|f(t)-f(s)|=0.$$

Exercise 11.2

Define $\mathscr{F}$ to be the collection of step functions with compact support in $\mathbb{R}_+$, that is, of functions f which can be written

$$f=\sum_{j=1}^{n}\lambda_j\mathbb{1}_{(t_{j-1},t_j]}$$

for constants λ_j. Write

$$\mathscr{E}^f:=e^{\int_0^T f(s)dW_s}.$$

Prove that the set $\left\{\mathscr{E}^f:f\in\mathscr{F}\right\}$ is total in $L^2\left(\Omega,\mathcal{F}_T,\mathbb{P}\right)$ where $\mathcal{F}_T=\bigvee_{s=0}^T\sigma(W_s)$ (that is, its span is dense).

Exercise 11.3

For any given $F\in L^2\left(\Omega,\mathcal{F}_T,\mathbb{P}\right)$, where $\mathcal{F}_T=\bigvee_{s=0}^T\sigma(W_s)$, prove that there exits a unique predictable process H such that

$$F=\mathbb{E}(F)+\int_0^T H_s dW_s.$$

Exercise 11.4

Let $T:\mathcal{X}\to\mathcal{X}$ be a measurable mapping, and $\xi:\Omega\to\mathcal{X}$ satisfy that $T\xi=\xi$ in law. Show that for every measurable function f, the discrete time process $(f(T^n\xi))_n$ is stationary.